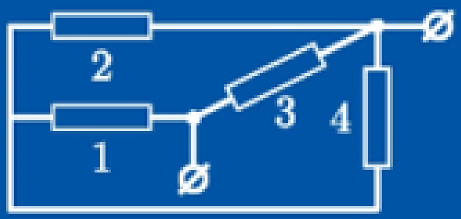
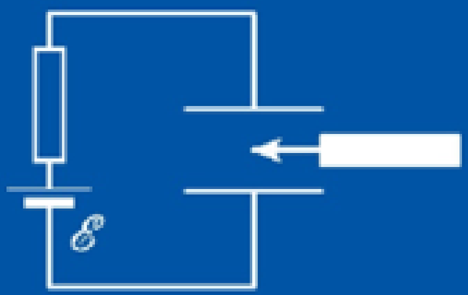
Moscow institute pf physics and technology
(state university)

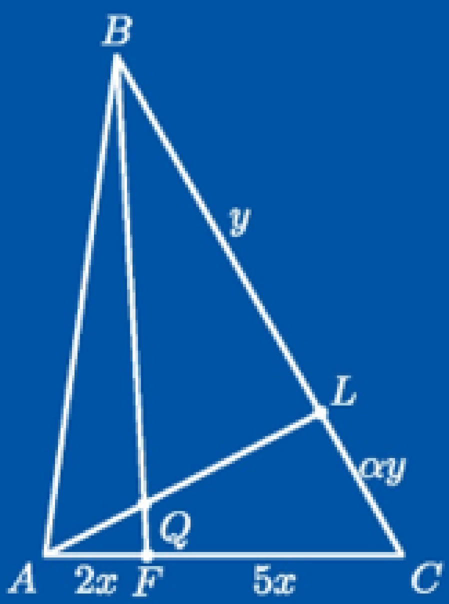


Problems

of the Phystech.International 2017

olympiad in physics and mathematics





(educational methodical appliance
in physics and mathematics)

PROBLEMS

of the “Phystech.International” 2017 olympiad in physics and mathematics

Educational methodical appliance

Moscow
MIPT
2018

UDC 53
BBK 22.3

P61 **Pronlems of the olympiad in physics and mathematics “Phystech.International” 2017.** (Educational methodical appliance). // Chivilev V., Ouskov V., Sheronov A., Yuriev Yu., Plis V., Agakhanov N., Glukhov I., Gorodetskiy S., Podlipskii O. M.: MIPT, 2018. – 45 c.

Here follows problems, offered at final stage of the olympiad “Phystech.International” on december, 2017. (2017–2018 school year).

All problems provided with answers, part of them—with detailed solutions. Each work was given 4,5 hours for completion.

Problems intended for matriculation students of MIPT and other technological universities, as well as for teachers of schools with advanced studying of physics and mathematics.

UDC 53
BBK 22.3

BBK means “Russian library and bibliographic classification” (BBK) index

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PHYSICS

Grade 9, Problem set #1

1. The first car of a train passes by an observer standing on the platform in $\tau_1 = 1$ s, and the second car in $\tau_2 = 1.5$ s. The length of a car is $L = 12$ m. Determine the train velocity V_0 at the beginning of the observation. The train is uniformly decelerating along a straight line.

2. A stone is thrown at some angle to the horizontal at initial velocity $V_0 = 10$ m/s. At $\tau = 0.5$ s after the throw the stone velocity has decreased to $V = 7$ m/s. Determine the moment T at which the stone reached the highest altitude. The free fall acceleration is $g = 10$ m/s^2.

3. A ball suspended on a thread has been given some initial horizontal velocity. The ball acceleration is horizontally directed when the thread is at the angle $\alpha = 30°$ to the vertical. Find the angle α_{max} the thread will make to the vertical at full swing.

4. A very light calorimeter contains $M = 0.1$ kg of water and a chunk of ice of mass $m = 0.05$ kg. The temperature of the water and ice is $t_1 = 0$ °C while the ambient temperature is $t_1 = 20$ °C. Due to poor insulation the ice is gradually melting down, so that $m_1 = 1$ g of ice melts into water in $\tau = 5$ min. Evaluate the time T elapsed between the complete meltdown of ice and the moment the water temperature has increased by $\Delta t = 1$ °C. The specific heat of fusion of ice is $\lambda = 3.3 \cdot 10^5$ J/kg and the specific heat of water is $c = 4200$ J/(kg·K).

5. The electric circuit shown in the diagram is connected to a power source of $U = 18$ V. Each resistor value is $r = 5\ \Omega$. Determine the power dissipated by resistor 1.

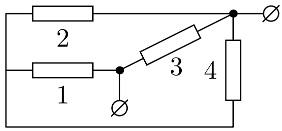

Fig. for problem 5

Grade 9, Problem set #2

1. The first car of a train passes by an observer standing on the platform in $\tau_1 = 1$ s, and the second car in $\tau_2 = 1.5$ s. The length of a car is $L = 12$ m. Determine the time T from the start of the observation up to full stop of the train. The train is uniformly decelerating along a straight line.

2. A stone is thrown at some angle to the horizontal at initial velocity $V_0 = 10$ m/s. At $\tau = 0.5$ s after the throw the stone velocity has decreased to $V = 7$ m/s. Determine the highest altitude H of the stone fly. The free fall acceleration is $g = 10$ m/s^2.

3. A ball suspended on a thread has been driven aside so that thread become strictly horizontal. Determine the angle α of the thread at the moment when acceleration of the ball directed horizontally.

4. A calorimeter contains $m_1 = 2$ kg of ice at temperature $t_1 = -5$ °C. Then water of mass $m_2 = 200$ g and temperature $t_2 = +5$ °C was added into the calorimeter. The temperature of the water and ice is $t_1 = 0$ °C while the ambient temperature is $t_1 = 20$ °C. Determine the mass m of ice in the calorimeter after establishing of thermal equilibrium. The specific heat of ice is $c_1 = 2100$ J/(kg·K), water $c_2 = 4200$ J/(kg·K). The specific heat of fusion of ice is $\lambda = 3.3 \cdot 10^5$ J/kg.

5. The electric circuit shown in the diagram is connected to a power source of direct voltage. All resistors are of equal value. The power dissipated by resistor 1 is $P_1 = 10$ W. Determine the power P dissipated by the whole circuit.

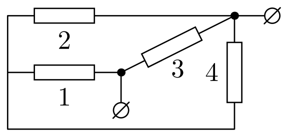

Fig. for problem 5

Grade 10, Problem set #3

1. A boy kicked the ball, which was lying on a horizontal surface at some distance from a vertical wall. The ball went flying at $\alpha = 30°$ to the horizontal, recoiled elastically from the wall, and landed exactly at the initial point in $t_0 = 1.5$ s after the kick.

1) What is the distance L between the ball and the wall?

2) Determine the height H at which the ball hit the wall.

Assume the free fall acceleration to be 10 m/s^2.

2. A ball of mass m_1, which has been sliding along a smooth horizontal surface, collides elastically head-on with a ball of mass m_2, which has been at rest on the same surface. After the collision, the ball of mass m_1 recoils in the opposite direction at $1/3$ of its initial velocity.

1) Determine the ratio $\frac{m_2}{m_1}$.

2) Determine the ratio of the velocity of the ball m_2 after the collision to the velocity of the ball m_1 before the collision.

3. A rectangular block is sliding on a smooth horizontal surface toward a ball sliding toward the block. The ball and the block are going along the same straight line. The ball velocity is perpendicular to the block face it is striking. The block mass is much greater than the ball mass. After the elastic collision, the ball is going in the opposite direction at the speed which is one half of its initial speed.

Determine the ratio of the velocities of the block and the ball before the collision.

4. Two thermally insulated containers of equal volume are connected by a short tube with a valve which is initially shut. The first container is filled with $\nu_1 = 1/3$ mol of a monoatomic ideal gas at $T_1 = 300$ K and the second one with $\nu_2 = 1/5$ mol of another monoatomic ideal gas at $T_2 = 500$ K. Then the valve is opened and the gases mix.

1) Determine the equilibrium temperature in the containers.

2) Determine the ratio of the final pressure of gas mixture to the initial pressure in the second container.

5. The volume of an ideal gas increases by the factor $n = 3$ in an isobaric process and then increases again by the same factor $n = 3$ in a process such that gas pressure P is directly proportional to its volume V.

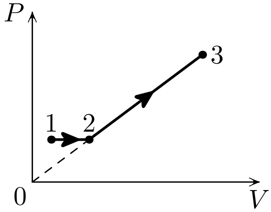

Fig. for problem 5

1) Determine the ratio of the final gas temperature to its initial temperature.

2) Determine the ratio of the work done by the gas during the isobaric process to the work done during the process in which its pressure P is directly proportional to its volume V.

Grade 10, Problem set #4

1. A boy kicked the ball, which was lying on a horizontal surface at some distance from a vertical wall. The ball went flying at $\alpha = 60°$ to the horizontal, recoiled elastically from the wall, and landed exactly at the initial point in $t_0 = 2$ s after the kick.

1) What is the distance L between the ball and the wall?

2) Determine the height H at which the ball hit the wall.

Assume the free fall acceleration to be 10 m/s^2.

2. A ball of mass m_1, which has been sliding along a smooth horizontal surface, collides elastically head-on with a ball of mass m_2, which has been at rest on the same surface. After the collision, the ball of mass m_1 recoils in the opposite direction at $1/2$ of its initial velocity.

1) Determine the ratio $\frac{m_2}{m_1}$.

2) Determine the ratio of the velocity of the ball m_2 to the velocity of the ball m_1 before the collision.

3. A rectangular block is sliding on a smooth horizontal surface toward a ball sliding toward the block. The ball and the block are going along the same straight line. The ball velocity is perpendicular to the block face it is striking. The block mass is much greater than the ball mass. After the elastic collision, the ball is going in the opposite direction at the speed which is 4 times greater its initial speed.

Determine the ratio of the velocities of the block and the ball before the collision.

4. Two thermally insulated containers of equal volume are connected by a short tube with a valve which is initially shut. The first container is filled with $\nu_1 = 1/2$ mol of a monoatomic ideal gas at $T_1 = 200$ K and the second one with $\nu_2 = 1/3$ mol of another monoatomic ideal gas at $T_2 = 300$ K. Then the valve is opened and the gases mix.

1) Determine the equilibrium temperature in the containers.

2) Determine the ratio of the final pressure of gas mixture to the initial pressure in the first container.

5. The volume of an ideal gas increases by the factor $n = 2$ in an isobaric process and then increases again by the same factor $n = 2$ in a process such that gas pressure P is directly proportional to its volume V.

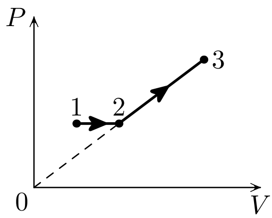

Fig. for problem 5

1) Determine the ratio of the final gas temperature to its initial temperature.

2) Determine the ratio of the work done by the gas during the isobaric process to the work done during the process in which its pressure P is directly proportional to its volume V.

Grade 11, Problem set #5

1. A small ball is suspended on a light thread 50 cm long. What minimal horizontal velocity should the ball be given in order to revolve a full circle in the vertical plane? Assume $g = 10$ m/s^2.

2. A small puck of mass m and a loose slide of mass $3m$ (see the Fig.) are on a smooth horizontal table. The puck is heading toward the slide at a speed v_0. Then the puck climbs the slide without friction, while remaining in contact with it all the way, and goes down in the opposite direction.

1) What is the maximum height reached by the puck?

2) What is the puck velocity when it leaves the slide?

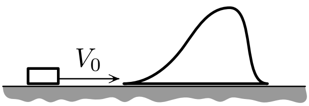

Fig. for problem 2

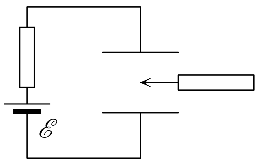

Fig. for problem 4

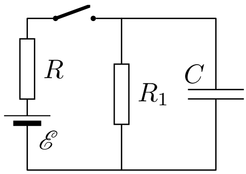

Fig. for problem 5

3. A thermally insulated container of the volume $V = 8.31 \cdot 10^{-3}$ m^3 is separated by a partition in two parts of different volumes. The first part is filled with $\nu_1 = 0.2$ mol of helium at 27 °C. The second part contains $\nu_2 = 0.3$ mol of helium at 7 °C. Then the partition ruptures.

1) What is the equilibrium temperature (in Celsius) in the container?

2) Determine the equilibrium pressure in the container.

4. A flat air-gap capacitor of capacitance C_0 is connected to a battery of emf $\mathcal{E}$ via a resistor (see the Fig.). Someone inserts an uncharged conducting plate into the gap parallel to the capacitor plates by placing it precisely against the plates. The plate shape coincides with the shape of capacitor plates. The plate thickness is one quarter of the air gap.

1) Determine the assembly capacitance.

2) What is the charge flowing through the resistor during the plate insertion?

5. The switch of the circuit shown in the diagram is initially opened. The parameters of the circuit elements are indicated. The internal resistance of the battery is combined with R and $R_1 = 3R$. Then the switch is closed and opened again when the circuit operates at a stationary regime. The values of C, $\mathscr{E}$, and R are assumed to be known.

1) Determine the current through the battery just after the switch is closed.

2) Determine the stationary voltage across the capacitor when the switch is closed.

3) What is the net heat released in the circuit after the switch was opened?

Grade 11, Problem set #6

1. A small ball is suspended on a light thread 18 cm long. What minimal horizontal velocity should the ball be given in order to revolve a full circle in the vertical plane? Assume $g = 10$ m/s^2.

2. A small coin of mass m and a loose slide of mass $4m$ (see the Fig.) are on a smooth horizontal table. The coin is heading toward the slide at a speed v_0. Then the coin climbs the slide without friction, while remaining in contact with it all the way, and goes down in the opposite direction.

1) What is the maximum height reached by the coin?

2) What is the coin velocity when it leaves the slide?

3. A thermally insulated container of the volume $V = 8.31 \cdot 10^{-3}$ m^3 is separated by a partition in two parts of different volumes. The first part is filled with $\nu_1 = 0.2$ mol of helium at 27 °C. The second part contains $\nu_2 = 0.3$ mol of helium at 7 °C. Then the partition rup-

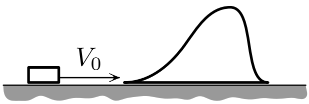

Fig. for problem 2

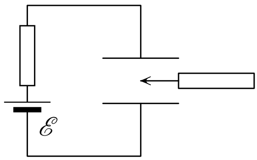

Fig. for problem 4

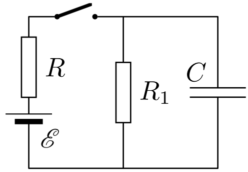

Fig. for problem 5

tures.

1) What is the equilibrium temperature (in Celsius) in the container?

2) Determine the equilibrium pressure in the container.

4. A flat air-gap capacitor of capacitance C_0 is connected to a battery of emf $\mathscr{E}$ via a resistor (see the Fig.). Someone inserts an uncharged conducting plate into the gap parallel to the capacitor plates by placing it precisely against the plates. The plate shape coincides with the shape of capacitor plates. The plate thickness is 1/3 of the air gap.

1) Determine the assembly capacitance.

2) What is the charge flowing through the resistor during the plate insertion?

5. The switch of the circuit shown in the diagram is initially opened. The parameters of the circuit elements are indicated. The internal resistance of the battery is combined with R and $R_1 = 4R$. Then the switch is closed and opened again when the circuit operates at a stationary regime. The values of C, $\mathscr{E}$, and R are assumed to be known.

1) Determine the current through the battery just after the switch is closed.

2) Determine the stationary voltage across the capacitor when the switch is closed.

3) What is the net heat released in the circuit after the switch was opened?

MATHEMATICS

Grade 9, Problem set #1

1. Parabola $y = 3x^2$ intersects with lines $y = 147$, $y = 75$ and $y = a$, thus forming a line segment on each of them. Find all values of parameter a such that these segments form a right triangle.

2. A quadrilateral $ABCD$ is given. Three circles ω_1, ω_2 and ω_3 that are pairwise tangent to each other are situated inside this quadrilateral and have equal radii. It is known that ω_1 touches sides AD and DC, ω_2 touches sides DC and CB, and ω_3 touches sides CB, BA and AD. Find the radii of the circles given that $AD+BC-AB-CD = 30$.

3. Cipollino wants to put all his stamps into a new album. If he puts 22 stamps per sheet, there will not be enough place for all the stamps. If he puts 26 stamps per sheet, then at least one sheet will be empty. If somebody presents Cipollino with the same album which has 21 stamps per sheet, Cipollino will own exactly 700 stamps. How many stamps does Cipollino have now? (All stamps have the same size.)

4. Find all values of parameter a such that solutions of the inequality $|ax - 3a| \leqslant \sqrt{x-1}$ form an interval with its length equal to 4.

5. How many 19-digit numbers are there that can be written with digits "2", "5" and "7" only (each of the digits is used at least once) and such that there are exactly eight digits "7" that are consecutive (i.e. one goes after another)?

6. Points F and L belong to sides AC and BC of triangle ABC respectively, and $AF : FC = 3 : 5$. Line segments BF and AL intersect at point Q; areas of triangles BQL and BAC are in the ratio of $4 : 25$. Find the distance from point L to line AC, if the distance from point Q to this line equals 12.

7. Pinocchio has chosen 5 integers from each of the intervals $[1; 25]$, $[26; 50]$, $[51; 75]$, $[76; 100]$. It turned out that for any two chosen numbers their difference is not a multiple of 25. Find the smallest possible value of the sum of all twenty numbers chosen by Pinocchio.

Grade 9, Problem set #2

1. Parabola $y = 5x^2$ intersects with lines $y = 125$, $y = 80$ and $y = a$, thus forming a line segment on each of them. Find all values of parameter a such that these segments form a right triangle.

2. A quadrilateral $ABCD$ is given. Three circles ω_1, ω_2 and ω_3 that are pairwise tangent to each other are situated inside this quadrilateral and have equal radii. It is known that ω_1 touches sides AD and DC, ω_2 touches sides DC and CB, and ω_3 touches sides CB, BA and AD. Find the radii of the circles given that $AD + BC - AB - CD = 28$.

3. Cipollino wants to put all his stamps into a new album. If he puts 15 stamps per sheet, there will not be enough place for all the stamps. If he puts 17 stamps per sheet, then at least one sheet will be empty. If somebody presents Cipollino with the same album which has 22 stamps per sheet, Cipollino will own exactly 900 stamps. How many stamps does Cipollino have now? (All stamps have the same size.)

4. Find all values of parameter a such that the solutions of the inequality $|ax - a| \leqslant \sqrt{x-3}$ form an interval with its length equal to 2.

5. How many 18-digit numbers are there that can be written with digits “3”, “5” and “8” only (each of the digits is used at least once) and such that there are exactly six digits “3” that are consecutive (i.e. one goes after another)?

6. Points F and L belong to sides AC and BC of triangle ABC respectively, and $AF : FC = 4 : 5$. Line segments BF and AL intersect at point Q; areas of triangles BQL and BAC are in the ratio of $1 : 25$. Find the distance from point L to line AC, if the distance from point Q to this line equals 12.

7. Pinocchio has chosen 6 integers from each of the intervals $[1; 40]$, $[41; 80]$, $[81; 120]$, $[121; 160]$. It turned out that for any two numbers their difference is not a multiple of 40. Find the smallest possible value of the sum of all twenty four numbers chosen by Pinocchio.

Grade 10, Problem set #3

1. Parabola $y = 2x^2 - 5x + 1$ intersects with lines $y = -1$, $y = 4$ and $y = a$, thus forming a line segment on each of them. Find all values of parameter a such that these segments form a right triangle.

2. How many 16-digit numbers are there that can be written with digits "3", "4" and "9" only (each of the digits is used at least once) and such that there are exactly four digits "9" that are consecutive (i.e. one goes after another)?

3. A quadrilateral $ABCD$ is given. Three circles ω_1, ω_2 and ω_3 that are pairwise tangent to each other are situated inside this quadrilateral and have equal radii. It is known that ω_1 touches sides AD and DC, ω_2 touches sides DC and CB, and ω_3 touches sides CB, BA and AD.

(a) Find the radii of the circles given that $AD + BC - AB - CD = 24$.

(b) Find angle AOB, where O is a center of ω_3.

4. Find all values of parameter a such that the solutions of the inequality $|ax - 2a| \leqslant \sqrt{x-1}$ form an interval with its length equal to 3.

5. Several workers can do some task in 28 days. If there had been two more workers and each of them had worked an extra hour a day, they would have done this task in 21 days. Had there been 4 more additional workers (i.e. 6 more workers compared with the initial situation) and they had worked for one more hour (i.e. two additional hours compared with the initial situation), they would have done this task in 15 days. How many workers were there? (All workers work with the same rate.)

6. Points F and L belong to sides AC and BC of triangle ABC respectively, and $AF : FC = 7 : 3$. Line segments BF and AL intersect at point Q; areas of triangles BQL and BAC are in the ratio of $7 : 36$. Find the distance from point L to line AC, if the distance from point Q to this line equals 3.

7. Pinocchio has chosen 6 integers from each of the intervals $[1; 30]$, $[31; 60]$, $[61; 90]$, $[91; 120]$. It turned out that for any two numbers their difference is not a multiple of 30. Find the largest possible value of the sum of all twenty four numbers chosen by Pinocchio.

Grade 10, Problem set #4

1. Parabola $y = 3x^2 - 4x + 2$ intersects with lines $y = 17$, $y = 1$ and $y = a$, thus forming a line segment on each of them. Find all values of parameter a such that these segments form a right triangle.

2. How many 20-digit numbers are there that can be written with digits “1”, “5” and “6” only (each of the digits is used at least once) and such that there are exactly ten digits “5” that are consecutive (i.e. one goes after another)?

3. A quadrilateral $ABCD$ is given. Three circles ω_1, ω_2 and ω_3 that are pairwise tangent to each other are situated inside this quadrilateral and have equal radii. It is known that ω_1 touches sides AD and DC, ω_2 touches sides DC and CB, and ω_3 touches sides CB, BA and AD.

(a) Find the radii of the circles given that $AD + BC - AB - CD = 38$.

(b) Find angle AOB, where O is a center of ω_3.

4. Find all values of parameter a such that the solutions of the inequality $|ax - a| \leqslant \sqrt{x-2}$ form an interval with its length equal to 1.

5. Several workers can do some task in 21 days. If there had been two more workers and each of them had worked an extra hour a day, they would have done this task in 15 days. Had there been 4 more additional workers (i.e. 6 more workers compared with the initial situation) and they had worked for one more hour (i.e. two additional hours compared with the initial situation), they would have done this task in 10 days. How many workers were there? (All workers work with the same rate.)

6. Points F and L belong to sides AC and BC of triangle ABC respectively, and $AF : FC = 2 : 7$. Line segments BF and AL intersect at point Q; areas of triangles BQL and BAC are in the ratio of $8 : 21$. Find the distance from point L to line AC, if the distance from point Q to this line equals 13.

7. Pinocchio has chosen 7 integers from each of the intervals $[1; 50]$, $[51; 100]$, $[101; 150]$, $[151; 200]$. It turned out that for any two numbers their difference is not a multiple of 50. Find the largest possible value of the sum of all twenty eight numbers chosen by Pinocchio.

Grade 11, Problem set #5

1. Parabola $y = 2x^2$ intersects with lines $y = 98$, $y = 18$ and $y = a$, thus forming a line segment on each of them. Find all values of parameter a such that these segments form a triangle with an angle of $120°$?

2. Find the largest and the smallest values of a function $g(x) = \sin 3x \cdot \sin 7x - \sin^2 x + \cos^2 5x + 4$.

3. How many 17-digit numbers are there that can be written with digits "0", "7" and "8" only (each of the digits is used at least once) and such that there are exactly seven digits "8" that are consecutive (i.e. one goes after another)?

4. A quadrilateral $ABCD$ is given. Three circles ω_1, ω_2 and ω_3 that are pairwise tangent to each other are situated inside this quadrilateral and have equal radii. It is known that ω_1 touches sides AD and DC, ω_2 touches sides DC and CB, and ω_3 touches sides CB, BA and AD.

(a) Find the radii of the circles given that $AD + BC - AB - CD = 12$.

(b) Find angle AOB, where O is a center of ω_3.

(c) Let it be additionally known that $AO \cdot BO = 58$. Find AB.

5. Solve the inequality $\log_{\sqrt{x+7}-x}(x+4) \geqslant 1$.

6. Points F and L belong to sides AC and BC of triangle ABC respectively, and $AF : FC = 2 : 5$. Line segments BF and AL intersect at point Q; areas of triangles BQL and BAC are in the ratio of $5 : 12$. Find the distance from point L to line AC, if the distance from point Q to this line equals 6.

7. Pinocchio has chosen 6 integers from each of the intervals $[1; 45]$, $[46; 90]$, $[91; 135]$, $[136; 180]$, $[181; 225]$. It turned out that for any two numbers their difference is not a multiple of 45. Find the smallest possible value of the sum of all thirty numbers chosen by Pinocchio.

Grade 11, Problem set #6

1. Parabola $y = x^2$ intersects with lines $y = 169$, $y = 64$ and $y = a$, thus forming a line segment on each of them. Find all values of parameter a such that these segments form a triangle with an angle of 120°?

2. Find the largest and the smallest values of a function $g(x) = \sin 5x \cdot \sin 9x - \sin^2 7x - \cos^2 x - 3$.

3. How many 18-digit numbers are there that can be written with digits “0”, “5” and “9” only (each of the digits is used at least once) and such that there are exactly six digits “5” that are consecutive (i.e. one goes after another)?

4. A quadrilateral $ABCD$ is given. Three circles ω_1, ω_2 and ω_3 that are pairwise tangent to each other are situated inside this quadrilateral and have equal radii. It is known that ω_1 touches sides AD and DC, ω_2 touches sides DC and CB, and ω_3 touches sides CB, BA and AD.

(a) Find the radii of the circles given that $AD + BC - AB - CD = 10$.

(b) Find angle AOB, where O is a center of ω_3.

(c) Let it be additionally known that $AO \cdot BO = 42$. Find AB.

5. Solve the inequality $\log_{\sqrt{x+3}-x}(x + 5) \geqslant 1$.

6. Points F and L belong to sides AC and BC of triangle ABC respectively, and $AF : FC = 3 : 4$. Line segments BF and AL intersect at point Q; areas of triangles BQL and BAC are in the ratio of $1 : 16$. Find the distance from point L to line AC, if the distance from point Q to this line equals 9.

7. Pinocchio has chosen 5 integers from each of the intervals $[1; 35]$, $[36; 70]$, $[71; 105]$, $[106; 140]$, $[141; 175]$. It turned out that for any two numbers their difference is not a multiple of 35. Find the smallest possible value of the sum of all twenty five numbers chosen by Pinocchio.

PHYSICS

Evaluation Criteria

The Final Stage of Phystech.International Olympiad
December 17, 2017.
Maximal total points for each problem: 10.

9-th Grade, Problem Sets 1 and 2

Prb.	Evaluation Criteria	$\sum$ pts
1.	Analytic expression for the length of the 1-st car via V_0 and τ_1	3
	Analytic expression for the length of the 2-nd car via V_0, τ_1, and τ_2	3
	Correct answer in analytic form	3
	Correct numerical answer	1
2.	All necessary equations are correctly written	4
	Correct answer in analytic form	4
	Correct numerical answer	2
3.	Newton's 2-nd law (for the angle α)	4
	Energy conservation law	3
	Correct answer in analytic form	2
	Correct numerical answer	1
4.	All equations are correctly written	6
	Correct answer in analytic form	3
	Correct numerical answer	1
5.	All equations are correctly written	6
	Correct answer in analytic form	3
	Correct numerical answer	1

10-th Grade, Problem Sets 3 and 4

Prb.	Evaluation Criteria	$\sum$ pts
1.	Correct answer in analytic form to the 1-st question	4
	Correct numerical answer to the 1-st question	1
	Correct answer in analytic form to the 2-nd question	4
	Correct numerical answer to the 2-nd question	1
2.	Law of conservation of momentum is correctly written	3

	Law of conservation of energy is correctly written	3
	Correct answer to the 1-st question	2
	Correct answer to the 2-nd question	2
3.	Transition to the reference frame of the block is performed. Or, laws of conservation of energy and momentum are written correctly for the final block mass	5
	Correct answer is obtained by using the block reference frame or by taking the block mass to	5
4.	Correct answer in analytic form to the 1-st question	4
	Correct numerical answer to the 1-st question	1
	Correct answer in analytic form to the 2-nd question	4
	Correct numerical answer to the 2-nd question	1
5.	Correct answer in analytic form to the 1-st question	4
	Correct numerical answer to the 1-st question	1
	Correct answer in analytic form to the 2-nd question	4
	Correct numerical answer to the 2-nd question	1

11-th Grade, Problem Sets 5 and 6

Prb.	Evaluation Criteria	$\sum$ pts
1.	Understanding that velocity at the upper point is non-zero	2
	All necessary equations are correctly written	4
	Answer in analytic form	3
	Numerical answer	1
2.	Answer to the 1-st question	5
	Answer to the 2-nd question	5
3.	Answer in analytic form to the 1-st question	5
	Numerical answer to the 1-st question	1
	Answer in analytic form to the 2-nd question	3
	Numerical answer to the 2-nd question	1
4.	Answer to the 1-st question	5
	Answer to the 2-nd question	5
5.	Answer to the 1-st question	4
	Answer to the 2-nd question	3
	Answer to the 3-d question	3

NOTE. Correct answers in analytic form could be obtained in different ways. Correct numerical answer could be only one!

PHYSICS

Grade 9, Problem set #1

1. Answer: $v_0 = \dfrac{\tau_2^2 + 2\tau_1\tau_2 - \tau_1^2}{\tau_1\tau_2(\tau_1+\tau_2)} L = \dfrac{1{,}5^2 + 2\cdot 1{,}5 - 1}{1{,}5\cdot 2{,}5}\cdot 12 =$ $= 13{,}6$ m/s.

Solution.

$$\left.\begin{aligned} &a<0;\quad L = v_0\tau_1 + \frac{a\tau_1^2}{2};\quad L - v_0\tau_1 = \frac{a\tau_1^2}{2};\\ &2L = v_0(\tau_1+\tau_2) + \frac{a(\tau_1+\tau_2)^2}{2};\ 2L - v_0(\tau_1+\tau_2) = \frac{a(\tau_1+\tau_2)^2}{2}\end{aligned}\right\}\Rightarrow$$

$\Rightarrow \dfrac{L - v_0\tau_1}{2L - v_0(\tau_1+\tau_2)} = \dfrac{\tau_1^2}{(\tau_1+\tau_2)^2}$; $L(\tau_1+\tau_2)^2 - v_0\tau_1(\tau_1 + \tau_2)^2 = 2L\tau_1^2 - v_0\tau_1^2(\tau_1+\tau_2)$; $v_0 = \dfrac{\tau_2^2 + 2\tau_1\tau_2 - \tau_1^2}{\tau_1\tau_2(\tau_1+\tau_2)} L =$ $= \dfrac{1{,}5^2 + 2\cdot 1{,}5 - 1}{1{,}5\cdot 2{,}5}\cdot 12 = 13{,}6$ m/s.

2. Answer: $T = \dfrac{v_0^2 + (g\tau)^2 - v^2}{2g^2\tau} = 0{,}76$ s.

Solution. $\left.\begin{aligned} v_x &= v_0\cos\alpha,\\ v_y &= v_0\sin\alpha - g\tau\end{aligned}\right\}\Rightarrow$

$$\Rightarrow \left.\begin{aligned} &v^2 = v_x^2 + v_y^2 = v_0^2\cos^2\alpha + v_0^2\sin^2\alpha + (g\tau)^2 - 2v_0\sin\alpha\cdot g\tau,\\ &v^2 = v_0^2 + (g\tau)^2 - 2v_0\sin\alpha\cdot g\tau \Rightarrow \sin\alpha = \frac{v_0^2 + (g\tau)^2 - v^2}{2v_0 g\tau}\end{aligned}\right\}.$$

$$T = \frac{v_0\sin\alpha}{g} = \frac{v_0^2 + (g\tau)^2 - v^2}{2g^2\tau} = \frac{10^2 + (10\cdot 0{,}5)^2 - 7^2}{2\cdot 10^2\cdot 0{,}5} = 0{,}76 \text{ s.}$$

3. Answer: $\cos\alpha_{\max} = \dfrac{2\cos^2\alpha - \sin^2\alpha}{2\cos\alpha} = \dfrac{5}{4\sqrt{3}} \approx 0{,}72$; $\alpha_{\max} \approx 43{,}8^\circ$.

Solution. T is a thread tension.

$$T\cos\alpha = mg;\quad T = \frac{mg}{\cos\alpha}; \qquad (1)$$

$$T - mg\cos\alpha = \frac{mv^2}{l}. \qquad (2)$$

$(1)\to(2):\ \dfrac{mg}{\cos\alpha} - mg\cos\alpha = \dfrac{mv^2}{l};$

$$v^2 = gl\left(\frac{1}{\cos\alpha} - \cos\alpha\right) = \frac{gl\sin^2\alpha}{\cos\alpha}\,. \qquad (3)$$

Conservation of energy: $$\frac{mv^2}{2} = mgl(\cos\alpha - \cos\alpha_{\max}). \qquad (4)$$

$$(3) \to (4): \quad \frac{gl\sin^2\alpha}{2\cos\alpha} = gl(\cos\alpha - \cos\alpha_{\max}).$$

$$\cos\alpha_{\max} = \cos\alpha - \frac{\sin^2\alpha}{2\cos\alpha} = \frac{2\cos^2\alpha - \sin^2\alpha}{2\cos\alpha} = \frac{2\cdot\frac{3}{4} - \frac{1}{4}}{2\cdot\frac{\sqrt{3}}{2}} =$$

$$= \frac{5}{4\sqrt{3}} \approx 0{,}72, \quad \alpha_{\max} \approx 43{,}8^\circ.$$

4. Answer: $T = \dfrac{(M+m)c\Delta t}{m_1\lambda}\tau \approx 9{,}5$ min.

Solution. Heat influx: $$\frac{\Delta Q}{\Delta T} = \frac{\lambda m_1}{\tau} \qquad (1)$$

$$\frac{\Delta Q}{\Delta T}T = (M+m)c\Delta t \qquad (2)$$

$$(1) \to (2): \quad \frac{\lambda m_1}{\tau}T = (M+m)c\Delta t.$$

$$T = \frac{(M+m)c\Delta t}{m_1\lambda}\tau = \frac{0{,}15\cdot 4200\cdot 1}{10^{-3}\cdot 3{,}3\cdot 10^5}\cdot 5 \approx 9{,}5 \text{ min}.$$

5. Answer: $P_1 = \dfrac{4V^2}{9r} = 28{,}8$ W.

Solution.

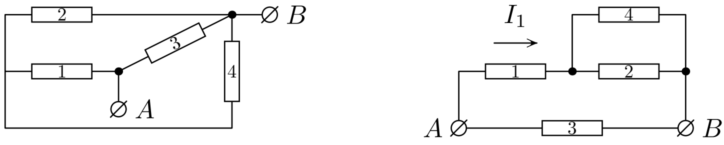

Fig. 1

$$r_{24} = \frac{r}{2}\,.$$

Resistance of the upper branch:

$$r_{124} = r + r_{24} = r + \frac{r}{2} = \frac{3}{2}r. \qquad (1)$$

$$I_1 = \frac{U}{r_{124}} = \frac{U}{3/2r} = \frac{2}{3}\frac{U}{r}\,. \tag{2}$$

$$P_1 = I_1^2 r = \frac{4V^2}{9r} = \frac{4\cdot 18^2}{9\cdot 5} = 28{,}8\ \text{W}.$$

Grade 9, Problem set #2

1. Answer: $T = \dfrac{\tau_2^2 + 2\tau_1\tau_2 - \tau_1^2}{2(\tau_2-\tau_1)} = 4{,}25\ \text{s}.$

Solution.

$$\left.\begin{aligned} a<0;\quad L = v_0\tau_1 + \frac{a\tau_1^2}{2};\quad L - \frac{a\tau_1^2}{2} = v_0\tau_1;\\ 2L = v_0(\tau_1+\tau_2) + \frac{a(\tau_1+\tau_2)^2}{2};\ 2L - \frac{a(\tau_1+\tau_2)^2}{2} = v_0(\tau_1+\tau_2)\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow \frac{L - a\tau_1^2/2}{2L - a(\tau_1+\tau_2)^2/2} = \frac{\tau_1}{\tau_1+\tau_2};\ L(\tau_1+\tau_2) - \frac{a\tau_1^2(\tau_1+\tau_2)}{2} =$$
$$= 2L\tau_1 - \frac{a(\tau_1+\tau_2)^2\tau_1}{2}\,.$$

$$a = \frac{2L(\tau_1-\tau_2)}{\tau_1\tau_2(\tau_1+\tau_2)}\,. \tag{1}$$

$$v_0 = \frac{L - a\tau_1^2/2)}{\tau_1} = \frac{L}{\tau_1} - \frac{a\tau_1}{2}\,. \tag{2}$$

$$T = \frac{v_0}{|a|} = \frac{v_0}{-a} = \frac{\tau_1}{2} - \frac{L\tau_1\tau_2(\tau_1+\tau_2)}{\tau_1\cdot 2L(\tau_1-\tau_2)};$$

$$T = \frac{\tau_2^2 + 2\tau_1\tau_2 - \tau_1^2}{2(\tau_2-\tau_1)} = \frac{1{,}5^2 + 2\cdot 1{,}5 - 1}{2\cdot 0{,}5} = 4{,}25\ \text{s}.$$

2. Answer: $H = \dfrac{(v_0^2 + (g\tau)^2 - v^2)^2}{8g^3\tau^2} \approx 2{,}9\ \text{m}.$

Solution. $$\left.\begin{aligned} v_x &= v_0\cos\alpha,\\ v_y &= v_0\sin\alpha - g\tau\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow \begin{aligned} v^2 &= v_x^2 + v_y^2 = v_0^2\cos^2\alpha + v_0^2\sin^2\alpha + (g\tau)^2 - 2v_0\sin\alpha\cdot g\tau,\\ v^2 &= v_0^2 + (g\tau)^2 - 2v_0\sin\alpha\cdot g\tau \Rightarrow \sin\alpha = \frac{v_0^2 + (g\tau)^2 - v^2}{2v_0 g\tau}\,.\end{aligned}$$

$$H = \frac{(v_0\sin\alpha)^2}{2g} = \frac{(v_0^2 + (g\tau)^2 - v^2)^2}{8g^3\tau^2} =$$
$$= \frac{(10^2 + (10\cdot 0{,}5)^2 - 7^2)^2}{/} 8\cdot 10^3\cdot 0{,}5^2 \approx 2{,}9\ \text{m}.$$

3. Answer: $\sin\alpha = \sqrt{\dfrac{2}{3}}$; $\alpha \approx 54{,}7^\circ$.

Solution. T is a thread tension.

$$T\cos\alpha = mg; \quad T = \frac{mg}{\cos\alpha}. \tag{1}$$

$$T - mg\cos\alpha = \frac{mv^2}{l}. \tag{2}$$

$$(1) \to (2):\ \frac{mg}{\cos\alpha} - mg\cos\alpha = \frac{mv^2}{l};$$

$$v^2 = gl\left(\frac{1}{\cos\alpha} - \cos\alpha\right) = \frac{gl\sin^2\alpha}{\cos\alpha}. \tag{3}$$

Conservation of energy: $$\frac{mv^2}{2} = mgl\cos\alpha. \tag{4}$$

$$(3) \to (4):\ \frac{gl\sin^2\alpha}{2\cos\alpha} = gl\cos\alpha; \quad \sin^2\alpha = 2\cos^2\alpha;$$

$$\sin^2\alpha = 2(1-\sin^2\alpha). \quad \sin\alpha = \sqrt{\frac{2}{3}}; \quad \alpha \approx 54{,}7^\circ.$$

4. Answer: $\Delta m_2 = \dfrac{m_1c_1(t_{пл} - t_1) - m_2c_2(t_2 - t_{пл})}{\lambda} =$
$= \dfrac{2\cdot 2100\cdot 5 - 0{,}2\cdot 4200\cdot 5}{3{,}3\cdot 10^5} \approx 0{,}051$ kg. $m = m_1 + \Delta m_2 \approx 2 +$
$+ 0{,}051 = 2{,}051$ kg.

Solution. Suppose, some water freezes. The heat released by water is:

$$Q_1 = m_2c_2(t_2 - t_{пл}) + \Delta m_2\lambda. \tag{1}$$

The heat received by ice is:

$$Q_2 = m_1c_1(t_{пл} - t_1). \tag{2}$$

Since $Q_1 = Q_2$, it follows from (1) and (2) that: $m_2c_2(t_2 - t_{пл}) +$
$+ \Delta m_2\lambda = m_1c_1(t_{пл} - t_1)$;

$$\Delta m_2 = \frac{m_1c_1(t_{пл} - t_1) - m_2c_2(t_2 - t_{пл})}{\lambda} =$$

$$= \frac{2\cdot 2100\cdot 5 - 0{,}2\cdot 4200\cdot 5}{3{,}3\cdot 10^5} \approx 0{,}051 \text{ kg}.$$

Mass of ice: $m = m_1 + \Delta m_2 \approx 2 + 0{,}051 = 2{,}051$ kg.

5. Answer: $P = \dfrac{15}{4}P_1 = 37{,}5$ W.

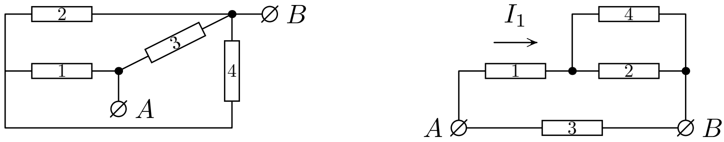

Fig. 2

Solution. U is a voltage across A and B. Resistance of the upper branch:

$$r_{124} = r + \frac{r}{2} = \frac{3}{2}r. \qquad (1)$$

$$r_{AB} = \frac{r_{124} \cdot r}{r_{124} + r} = \frac{\frac{3}{2}\,r^2}{\frac{3}{2}\,r + r} = \frac{3}{5}r. \qquad (2)$$

$$I_1 = \frac{U}{r_{124}} = \frac{U}{\frac{3}{2}\,r} = \frac{2}{3}\frac{U}{r}\,. \qquad (3)$$

$$\left.\begin{aligned} P_1 &= I_1^2 r = \frac{4}{9}\frac{U^2}{r} \\ P &= \frac{U^2}{r_{AB}} = \frac{5U^2}{3r} \end{aligned}\right\} \Rightarrow \frac{P_1}{P} = \frac{4}{9}\cdot\frac{3}{5} = \frac{4}{15};$$

$$P = \frac{15}{4}P_1 = \frac{15}{4}\cdot 10 = 37{,}5 \text{ W}.$$

Grade 10, Problem set #3

1. Answer: 1) $L = \dfrac{g t_0^2 \operatorname{ctg}\alpha}{4} \approx 9{,}7$ m; 2) $H = \dfrac{g t_0^2}{8} \approx 2{,}8$ m.

Solution. Let v be the ball speed just after it was kicked. In terms of the required parameters, the ball equations of motion in the vertical and horizontal directions are:

$$v\sin\alpha = \frac{g t_0}{2}; \quad v\cos\alpha \cdot \frac{t_0}{2} = L; \quad H = \frac{(v\sin\alpha)^2}{2g}\,.$$

Using the first two equations, one obtains:

$$L = \frac{g t_0^2 \operatorname{ctg}\alpha}{4} \approx 9{,}7 \text{ m}.$$

The first and the third equations yield

$$H = \frac{gt_0^2}{8} \approx 2{,}8 \text{ m}.$$

2. Answer: 1) $\frac{m_2}{m_1} = 2$; 2) $\frac{V_2}{V_0} = \frac{2}{3}$.

Solution. Let V_0 be the velocity of the ball of mass m_1 before the collision V_1 and V_2 be the velocities of the balls of masses m_1 and m_2 after the collision. According to the laws of conservation of energy and momentum,

$$\frac{m_1V_0^2}{2} = \frac{m_1V_1^2}{2} + \frac{m_2V_2^2}{2}, \quad m_1V_0 = m_1V_1 + m_2V_2.$$

From these equations, one obtains:

$$V_1 = V_0\frac{m_1 - m_2}{m_1 + m_2}; \qquad V_2 = \frac{2m_1V_0}{m_1 + m_2}.$$

Obviously, the ball of mass m_1 goes backwards if $m_2 > m_1$.

1) According to the problem statement, $V_1 = V_0\frac{m_1 - m_2}{m_1 + m_2} = -\frac{V_0}{3}$. Hence, $\frac{m_2}{m_1} = 2$.

2) Using the second equation and the ratio $\frac{m_2}{m_1} = 2$, one obtains $\frac{V_2}{V_0} = \frac{2}{3}$.

3. Answer: $\frac{V}{U} = 2$.

Solution. Let V and U be the velocities of the ball and the block in the laboratory frame (LF). Consider a motion in the inertial frame of the block. In this frame the block is at rest and the ball is moving towards the block at a speed of $V+U$. After an elastic collision with a heavy block, the ball goes in the opposite direction at the speed of $V+U$. In the LF the ball velocity after the collision is $V+U+U = V+2U$. According to the problem statement, $V+2U = 2V$. Therefore, $\frac{V}{U} = 2$.

4. Answer: 1) $T = \frac{\nu_1T_1 + \nu_2T_2}{\nu_1 + \nu_2} = 375$ K;

2) $\frac{P}{P_2} = \frac{1}{2}\left(1 + \frac{\nu_1T_1}{\nu_2T_2}\right) = 1$.

Solution. Let T and P be the final temperature and pressure of the gas mixture, and P_1, P_2, and V be the initial pressure and volume of the gases contained.

1) The temperature T can be found from the conservation of the net internal energy of the gases:
$$\nu_1 \frac{3}{2} RT_1 + \nu_2 \frac{3}{2} RT_2 = (\nu_1 + \nu_2)\frac{3}{2} RT.$$
Therefore, $T = \dfrac{\nu_1 T_1 + \nu_2 T_2}{\nu_1 + \nu_2} = 375$ K.

2) Equations of state of the gases are:
$$P_1 V = \nu_1 RT_1, \quad P_2 V = \nu_2 RT_2, \quad P \cdot 2V = (\nu_1 + \nu_2)RT.$$
Using the expression for the temperature T, one obtains:
$$\frac{P}{P_2} = \frac{1}{2}\left(1 + \frac{\nu_1 T_1}{\nu_2 T_2}\right) = 1.$$

5. Answer: 1) $\dfrac{T_3}{T_1} = n^3 = 27$, 2) $\dfrac{A_{12}}{A_{23}} = \dfrac{2}{n(n+1)} = \dfrac{1}{6}$.

Solution. According to the problem statement, on the isobar, $P_1 = P_2$ and $V_2 = nV_1$. In the process 2–3: $P_3 = nP_2 = nP_1$ and
$$V_3 = nV_2 = n^2 V_1.$$

1) Using $\dfrac{P_1 V_1}{T_1} = \dfrac{P_3 V_3}{T_3}$ and the above equations, one obtains:
$$T_3 = n^3 T_1 = 27T_1.$$

2) The work done during the isobaric process is
$$A_{12} = P_1(V_2 - V_1) = (n-1)P_1 V_1.$$
The work done in the process 2–3 is
$$A_{23} = \frac{1}{2}(P_2 + P_3)(V_3 - V_2) = \frac{1}{2} n(n+1)(n-1)P_1 V_1.$$
Finally, $\dfrac{A_{12}}{A_{23}} = \dfrac{2}{(n+1)n} = \dfrac{1}{6}$.

Grade 10, Problem set #4

1. Answer: 1) $L = \dfrac{g t_0^2 \operatorname{ctg}\alpha}{4} \approx 5{,}8$ m; 2) $H = \dfrac{g t_0^2}{8} \approx 5$ m.

Solution. Let v be the ball speed just after it was kicked. The ball equations of motion in the vertical and horizontal directions in terms of the required parameters are

$$v\sin\alpha = \frac{gt_0}{2}; \quad v\cos\alpha \cdot \frac{t_0}{2} = L; \quad H = \frac{(v\sin\alpha)^2}{2g}.$$

Using the first two equations, one finds

$$L = \frac{gt_0^2 \operatorname{ctg}\alpha}{4} \approx 5{,}8 \text{ m}.$$

The first and the third equations yield

$$H = \frac{gt_0^2}{8} \approx 5 \text{ m}.$$

2. Answer: 1) $\frac{m_2}{m_1} = 3$; 2) $\frac{V_2}{V_0} = \frac{1}{2}$.

Solution. Let V_0 be the velocity of the ball of mass m_1 before the collision, V_1 and V_2 be the velocities of the balls of masses m_1 and m_2 after the collision. According to the laws of energy and momentum conservation,

$$\frac{m_1V_0^2}{2} = \frac{m_1V_1^2}{2} + \frac{m_2V_2^2}{2}, \quad m_1V_0 = m_1V_1 + m_2V_2.$$

From these equations, one finds:

$$V_1 = V_0\frac{m_1 - m_2}{m_1 + m_2}; \quad V_2 = \frac{2m_1V_0}{m_1 + m_2}.$$

Obviously, the ball of mass m_1 goes backwards if $m_2 > m_1$.

1) According to the problem statement, $V_1 = V_0\frac{m_1 - m_2}{m_1 + m_2} = -\frac{V_0}{2}$. Hence, $\frac{m_2}{m_1} = 3$.

2) Using the second equation and the ratio $\frac{m_2}{m_1} = 3$, one obtains $\frac{V_2}{V_0} = \frac{1}{2}$.

3. Answer: $\frac{V}{U} = \frac{2}{3}$.

Solution. Let the ball speed in the laboratory frame (LF) be V, and the block velocity be U. Consider a motion in the inertial frame of the block. In this frame the block is at rest and the ball is moving towards the block at a speed of $V+U$. After an elastic collision with a heavy block, the ball goes in the opposite direction at the speed of $V+U$. In the LF the ball velocity after the collision is $V+U+U =$

$= V + 2U$. According to the problem statement, $V + 2U = 4V$. The last equation yields $V/U = 2/3$.

4. Answer: 1) $T = \frac{\nu_1 T_1 + \nu_2 T_2}{\nu_1 + \nu_2} = 240$ K;

2) $\frac{P}{P_1} = \frac{1}{2}\left(1 + \frac{\nu_2 T_2}{\nu_1 T_1}\right) = 1$.

Solution. Let T and P be the final temperature and pressure of the gas mixture, P_1, P_2, and V be the initial pressure and volume of the gases contained.

1) The temperature T can be found from the conservation of the net internal energy of the gases:

$$\nu_1 \frac{3}{2} R T_1 + \nu_2 \frac{3}{2} R T_2 = (\nu_1 + \nu_2) \frac{3}{2} R T.$$

Therefore, $T = \frac{\nu_1 T_1 + \nu_2 T_2}{\nu_1 + \nu_2} = 240$ K.

2) Equations of state of the gases are:

$$P_1 V = \nu_1 R T_1, \quad P_2 V = \nu_2 R T_2, \quad P \cdot 2V = (\nu_1 + \nu_2) R T.$$

Using the expression for temperature T, one obtains:

$$\frac{P}{P_1} = \frac{1}{2}\left(1 + \frac{\nu_2 T_2}{\nu_1 T_1}\right) = 1.$$

5. Answer: 1) $\frac{T_3}{T_1} = n^3 = 8$, 2) $\frac{A_{12}}{A_{23}} = \frac{2}{n(n+1)} = \frac{1}{3}$.

Solution. According to the problem statement, on the isobar, $P_1 = P_2$ and $V_2 = nV_1$. In the process 2–3: $P_3 = nP_2 = nP_1$ and

$$V_3 = nV_2 = n^2 V_1.$$

1) Using $\frac{P_1 V_1}{T_1} = \frac{P_3 V_3}{T_3}$ and the above equations, one obtains:

$$T_3 = n^3 T_1 = 8T_1.$$

2) The work done during the isobaric process is

$$A_{12} = P_1(V_2 - V_1) = (n-1) P_1 V_1.$$

The work done in the process 2–3 is

$$A_{23} = \frac{1}{2}(P_2 + P_3)(V_3 - V_2) = \frac{1}{2} n(n+1)(n-1) P_1 V_1.$$

Finally, $\frac{A_{12}}{A_{23}} = \frac{2}{(n+1)n} = \frac{1}{3}$.

Grade 11, Problem set #5

1. Answer: $V = \sqrt{5gl} = 5$ m/s.

Solution. Let m be the ball mass, l be the thread length, V be the minimum velocity, and V_1 be the velocity at the upper point. The tension at the upper point vanishes. According to the Newton's 2-nd law, at the upper point $0 + mg = \frac{mV_1^2}{l}$. According to the law of energy conservation, $\frac{mV^2}{2} = \frac{mV_1^2}{2} + mg2l$. Therefore, $V = \sqrt{5gl} = 5$ m/s.

2. Answer: 1) $H = \frac{3}{8}\frac{V_0^2}{g}$; 2) $V = \frac{1}{2}V_0$.

Solution. 1) Let V_1 be the velocity of the puck and the slide at the maximum height H. According to the laws of conservation of energy and momentum, $mV_0 = (3m+m)V_1$, $\frac{1}{2}mV_0^2 = \frac{1}{2}(3m+m)V_1^2 + mgH$. Hence, $H = \frac{3}{8}\frac{V_0^2}{g}$.

2) Let u be the slide velocity after the puck has left. According to the laws of conservation of energy and momentum, $mV_0 = 3mu - mV$, $\frac{1}{2}mV_0^2 = \frac{1}{2}3mu^2 + \frac{1}{2}mV^2$. Hence, $V = \frac{1}{2}V_0$.

3. Answer: 1) $t = 15\,°\mathrm{C}$; 2) $P = 1{,}44 \cdot 10^5$ Pa.

Solution. 1) The net internal energy of the gases is conserved: $\nu_1 \frac{3}{2}RT_1 + \nu_2 \frac{3}{2}RT_2 = (\nu_1 + \nu_2)\frac{3}{2}RT$. Here, $T_1 = 300$ K and $T_2 = 280$ K. Therefore,

$$T = \frac{\nu_1 T_1 + \nu_2 T_2}{\nu_1 + \nu_2} = 288\ \mathrm{K}\ (15\,°\mathrm{C}).$$

2) $PV = (\nu_1 + \nu_2)RT$. Using the expression for T, one obtains

$$P = \frac{\nu_1 T_1 + \nu_2 T_2}{V}R = 1{,}44 \cdot 10^5\ \mathrm{Pa}.$$

4. Answer: 1) $C = \frac{4}{3}C_0$; 2) $q = \frac{1}{3}C_0\mathscr{E}$.

Solution. Let a distance between the plates be d and the plate area be S.

1) $C_0 = \dfrac{\mathscr{E}_0 S}{d}$ and $C = \dfrac{\mathscr{E}_0 S}{3d/4}$. Then, $C = \dfrac{4}{3}\, C_0$.

2) $q = C\mathscr{E} - C_0\mathscr{E} = \dfrac{1}{3}\, C_0\mathscr{E}$.

5. Answer: 1) $I_0 = \dfrac{\mathscr{E}}{R}$; 2) $T = \dfrac{3}{4}\,\mathscr{E}$; 3) $Q = \dfrac{9}{32}\, C\mathscr{E}^2$.

Solution. 1) The current via R_1 right after the switch was closed, was zero. The current through the battery is $I_0 = \dfrac{\mathscr{E}}{R}$.

2) In steady state the current through capacitor vanishes. The voltage across the capacitor equals to that across R_1 : $U = \dfrac{\mathscr{E}}{R + 3R}\, 3R = \dfrac{3}{4}\,\mathscr{E}$.

3) $Q = \dfrac{1}{2}\, CU^2 = \dfrac{9}{32}\, C\mathscr{E}^2$.

Grade 11, Problem set #6

Answer: $V = \sqrt{5gl} = 3$ m/s.

Solution. Let m be the ball mass, l be the thread length, V be the minimum velocity, and V_1 be the velocity at the upper point. The tension at the upper point vanishes. According to the Newton's 2-nd law, at the upper point $0 + mg = \dfrac{mV_1^2}{l}$. According to the law of energy conservation, $\dfrac{mV^2}{2} = \dfrac{mV_1^2}{2} + mg2l$. Therefore, $V = \sqrt{5gl} = 3$ m/s.

2. Answer: 1) $H = \dfrac{2}{5}\,\dfrac{V_0^2}{g}$; 2) $V = \dfrac{3}{5}\, V_0$.

Solution. 1) Let V_1 be the velocity of the coin and the slide at the maximum height H . According to the laws of conservation of energy and momentum, $mV_0 = (4m + m)V_1$, $\dfrac{1}{2}\, mV_0^2 = \dfrac{1}{2}\,(4m + m)V_1^2 + mgH$. Hence, $H = \dfrac{2}{5}\,\dfrac{V_0^2}{g}$.

2) Let u be the slide velocity after the coin has left it. According to the laws of conservation of energy and momentum, $mV_0 = 4mu - mV$, $\dfrac{1}{2}\, mV_0^2 = \dfrac{1}{2}\, 4mu^2 + \dfrac{1}{2}\, mV^2$. Hence, $V = \dfrac{3}{5}\, V_0$.

3. Answer: 1) $t = 31\ ^\circ\mathrm{C}$; 2) $P = 1{,}52 \cdot 10^5$ Pa.

Solution. 1) The net internal energy of the gases is conserved: $\nu_1 \frac{3}{2} RT_1 + \nu_2 \frac{3}{2} RT_2 = (\nu_1 + \nu_2) \frac{3}{2} RT$. Here, $T_1 = 400$ K and $T_2 = 280$ K. Therefore,

$$T = \frac{\nu_1 T_1 + \nu_2 T_2}{\nu_1 + \nu_2} = 304\ \mathrm{K}\ (31\ ^\circ\mathrm{C}).$$

2) $PV = (\nu_1 + \nu_2)RT$. Using the expression for T, one obtains:

$$P = \frac{\nu_1 T_1 + \nu_2 T_2}{V} R = 1{,}52 \cdot 10^5\ \Pi\mathrm{a}.$$

4. Answer: 1) $C = \frac{3}{2} C_0$; 2) $q = \frac{1}{2} C_0 \mathscr{E}$.

Solution. Let d and S be a distance between the plates and the plate area, respectively.

1) $C_0 = \frac{\mathscr{E}_0 S}{d}$, $C = \frac{\mathscr{E}_0 S}{2d/3}$. Hence, $C = \frac{3}{2} C_0$.

2) $q = C\mathscr{E} - C_0\mathscr{E} = \frac{1}{2} C_0 \mathscr{E}$.

5. Answer: 1) $I_0 = \frac{\mathscr{E}}{R}$; 2) $U = \frac{4}{5}\mathscr{E}$; 3) $Q = \frac{8}{25} C\mathscr{E}^2$.

Solution. 1) The current via R_1, right after the switch was closed, was zero. The current through the battery is $I_0 = \frac{\mathscr{E}}{R}$.

2) In steady state the current through capacitor vanishes. The voltage across the capacitor equals to that across R_1: $U = \frac{\mathscr{E}}{R + 4R} 4R = \frac{4}{5}\mathscr{E}$.

3) $Q = \frac{1}{2} CU^2 = \frac{8}{25} C\mathscr{E}^2$.

MATHEMATICS

Grade 9, Problem set #1

1. Answer: 72; 222.

Solution. We start with finding intersection points of lines with parabola (only positive values of a are considered since otherwise the parabola does not form a segment on the third line):

$$\begin{cases} y = 3x^2, \\ y = 147 \end{cases} \iff \begin{cases} x = \pm 7, \\ y = 147; \end{cases} \quad \begin{cases} y = 3x^2, \\ y = 75 \end{cases} \iff \begin{cases} x = \pm 5, \\ y = 75; \end{cases}$$

$$\begin{cases} y = 3x^2, \\ y = a \end{cases} \iff \begin{cases} x = \pm\sqrt{\dfrac{a}{3}}, \\ y = a. \end{cases}$$

Therefore, lengths of the segments in question are equal to 14, 10 and $\sqrt{\dfrac{4a}{3}}$. The right angle is the largest angle of a triangle and it lies opposite its largest side. Hence two cases are possible.

1) The right angle lies opposite the side that is equal to $\sqrt{\dfrac{4a}{3}}$. Then Pythagorean theorem yields $\dfrac{17+8a}{4} = \dfrac{9}{4} + \dfrac{49}{4}$, thus $a = \dfrac{41}{8}$.

2) The right angle is opposite the side that is equal to 14. Then Pythagorean theorem yields $196 = 100 + \dfrac{4a}{3}$, and so $a = 72$.

2. Answer: 15.

Solution. (a) Let lines DA and CB intersect at point M (see fig. 1). Triangle CMD is regular, as radii of all three circles are equal to each other. Let us designate radii of the circles as r, and distances from point C to the points where circle ω_2 touches the sides of triangle CMD as x (these distances are equal as segments of tangent lines drawn to a circle from one point). Then distances from point D to the points where circle ω_1 touches the sides of triangle CMD are also equal to x (due to the triangle being equilateral), and distances between the tangent points of any side of triangle CMD with the circles are equal to $2r$. Let distances from vertex A to the tangent points of the sides of the quadrilateral with circle ω_3 be equal to a, and distances from point B to tangent points of the sides of the quadrilateral with ω_3 be equal to b. Then the given equality can be written as $(x + 2r + a) + (x + 2r + b) - (a + b) - (2x + 2r) = 30$; hence $r = 15$.

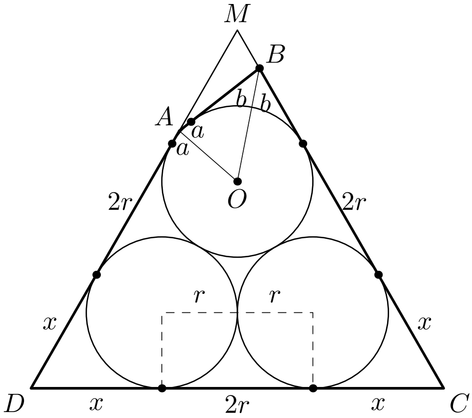

Fig. 1

3. Answer: 364.

Solution. Let the number of sheets in the album be equal to p, and a total number of stamps be equal to k. Then if we have 22 stamps per sheet, there are $22p$ stamps in total, and it is less than the number of stamps Cipollino has, so $22p < k$.

If we distribute 26 stamps per sheet, then at least one of the sheets is empty. It implies that the number of stamps in Cipollino's possession does not exceed $26(p-1)$, therefore $k \leqslant 26(p-1)$.

Finally, from the last condition we get that $k + 21p = 700$, and $k = 700 - 21p$. Substituting it into the inequalities above yields:

$$\begin{cases} 22p < 700 - 21p, \\ 700 - 21p \leqslant 26(p-1) \end{cases} \iff \begin{cases} p < \dfrac{700}{43}, \\ p \geqslant \dfrac{726}{47}. \end{cases}$$

There is only one integer value of p that satisfies both inequalities, and it is $p = 16$. Thus Cipollino has $700 - 21 \cdot 16 = 364$ stamps.

4. Answer: $\pm 2\sqrt{\sqrt{2}+1}$.

Solution. Both sides of the inequality are nonnegative, and so we can raise them to the second power. We get $(ax-3a)^2 \leqslant x-1$. As this inequality states that the former radicand is greater or equal than some exact square, it is equivalent to the initial one. Combining like terms yields $a^2x^2-\left(6a^2+1\right)x+\left(9a^2+1\right) \leqslant 0$.

If $a=0$, then $x \geqslant 1$, and that does not satisfy the condition. For all other values of a this inequality is quadratic, the graph of its left side being a parabola that is open up. For negative values of discriminant there are no solutions; if discriminant is zero, there is exactly one solution, and in case discriminant is negative the solution set is a closed interval between the roots of the quadratic function.

Discriminant D is equal to $\left(6a^2+1\right)^2-4a^2\left(9a^2+1\right)=8a^2+$ $+1$. The roots are given by $\dfrac{6a^2+1\pm\sqrt{D}}{2a^2}$, and the distance between them is $\dfrac{\sqrt{D}}{a^2}$. This distance has to be equal to 4, hence $\sqrt{D}=4a^2$, $D=16a^4$, $16a^4-8a^2-1=0$, $a=\pm\frac{1}{2}\sqrt{\sqrt{2}+1}$.

5. Answer: 24552.

Solution. There are twelve possibilities to place the sevens. After the spaces for them have been chosen, we have to fill the remaining eleven positions with twos and fives. For each of the spaces we can choose either a two or a five, hence there are 2^{11} ways possible. Yet two of them are to be excluded (when all 11 digits are identical, as each of the digits has to be used at least once). Therefore, the total amount is $12\left(2^{11}-2\right)=24552$.

6. Answer: 20.

Solution. Let us designate $CL:LB=\alpha$ (see fig. 2), and area of triangle ABC as S. Menelaus's theorem for triangle ACL and transversal BF yields $\dfrac{AF}{FC}\cdot\dfrac{CB}{BL}\cdot\dfrac{LQ}{QA}=1$, and from here follows $\dfrac{3}{5}\cdot(\alpha+1)\cdot\dfrac{LQ}{QA}=1$, $\dfrac{LQ}{QA}=\dfrac{5}{3\alpha+3}$.

Triangles ALB and ACB have a common altitude dropped from vertex A, and so their areas are in the same ratio as their corresponding bases, i.e. $S_{\triangle ALB}=\dfrac{S}{\alpha+1}$.

Triangles ALB and QLB have a common altitude dropped from vertex B, hence their areas are also in the same ratio as their bases; consequently $S_{\triangle BLQ} : S_{\triangle BLA} = \dfrac{QL}{AL} = \dfrac{5}{3\alpha+8}$. Therefore,

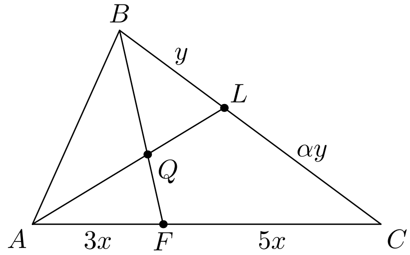

Fig. 2

$S_{\triangle BLQ} = \dfrac{5S}{(3\alpha+8)(\alpha+1)}$. It is given that the latter area is equal to $\dfrac{4S}{25}$ which implies that $4(3\alpha+8)(\alpha+1) = 125$, and so $\alpha = \dfrac{3}{2}$, the second root of this equation being negative.

Ratio of distances from points L and Q to line AC is equal to $LA : QA = (8+3\alpha) : (3+3\alpha) = 5 : 3$. Therefore, the distance in question is $12 \cdot \dfrac{5}{3} = 20$.

7. Answer: 960.

Solution. Let $d_1, \ldots, d_5$ be numbers from the first interval, d_6, $\ldots$, d_{10} be numbers from the second interval, $d_{11}, \ldots, d_{15}$ be numbers from the third interval and so on.

Let us notice that each number from the second interval can be represented as $d_i = 25 + c_i$ where $1 \leqslant c_i \leqslant 25$, each number from the third interval can be represented as $d_i = 50 + c_i$ where $1 \leqslant c_i \leqslant 25$ etc. Let us also designate $c_1 = d_1, \ldots, c_5 = d_5$.

Taking in account all the designations above, the sum considered is equal to $5 \cdot (25 + 50 + 75) + c_1 + c_2 + \ldots + c_{20} = 750 + c_1 + c_2 + \ldots + c_{20}$. Let us also notice that all numbers $c_1, c_2, \ldots, c_{20}$ have to be different (if $c_i = c_j$ then difference $d_i - d_j$ is a multiple of 25, and if $c_i \neq c_j$ then $d_i - d_j$ is not a multiple of 25). Therefore, sum of numbers d_i reaches its minimum, if c_i take values from 1 to 20 (in arbitrary order); this minimum is equal to $750 + 1 + 2 + \ldots + 20 = 750 + \dfrac{1+20}{2} \cdot 20 = 960$.

Grade 9, Problem set #2

1. Answer: 205; 45.

2. Answer: 14.

3. Answer: 372.

4. Answer: $\pm\sqrt{\dfrac{\sqrt{5}-2}{2}}$.

5. Answer: 53222.

6. Answer: 15.

7. Answer: 1740.

Grade 10, Problem set #3

1. Answer: $a=\dfrac{23}{8}$, $a=\dfrac{41}{8}$.

Solution. We start with finding intersection points of lines with parabola (we consider only such values of a that discriminant of a quadratic equation in the third system is positive since otherwise the parabola does not form a segment on the third line):

$$\begin{cases} y=2x^2-5x+1, \\ y=-1 \end{cases} \Longleftrightarrow \begin{cases} x=2 \text{ or } x=\dfrac{1}{2}, \\ y=-1; \end{cases}$$

$$\begin{cases} y=2x^2-5x+1, \\ y=4 \end{cases} \Longleftrightarrow \begin{cases} x=3 \text{ or } x=-\dfrac{1}{2}, \\ y=4; \end{cases}$$

$$\begin{cases} y=2x^2-5x+1, \\ y=a \end{cases} \Longleftrightarrow \begin{cases} x=\dfrac{5\pm\sqrt{17+8a}}{4}, \\ y=a. \end{cases}$$

Therefore, lengths of the segments in question are equal to $\dfrac{3}{2}$, $\dfrac{7}{2}$ and $\dfrac{\sqrt{17+8a}}{2}$. The right angle is the largest angle in a triangle and it lies opposite its largest side. Hence two cases are possible.

1) The right angle lies opposite the side equal to $\dfrac{\sqrt{17+8a}}{2}$. Then Pythagorean theorem yields $\dfrac{17+8a}{4}=\dfrac{9}{4}+\dfrac{49}{4}$, thus $a=\dfrac{41}{8}$.

2) The right angle is opposite the side equal to $\frac{7}{2}$. Then Pythagorean theorem yields $\frac{49}{4} = \frac{17+8a}{4} + \frac{9}{4}$, and so $a = \frac{23}{8}$.

2. Answer: 53222.

Solution. There are thirteen possibilities to place the nines. After spaces for them have been chosen, we have to fill the remaining twelve positions with threes and fours. For each of the spaces we can choose either a three or a four, hence there are 2^{12} ways possible. Yet two of them are to be excluded (when all 12 digits are identical, as each of the digits has to be used at least once). Therefore, the total amount is $13\left(2^{12}-2\right) = 53222$.

3. Answer: (a) 12; (b) $60°$.

Solution. (a) Let lines DA and CB intersect at point M (see fig. 1, p. 32). Triangle CMD is regular, as radii of all three circles are equal to each other. Let us designate radii of the circles as r, and distances from point C to the points where circle ω_2 touches the sides of triangle CMD as x (these distances are equal as segments of tangent lines drawn to a circle from one point). Then distances from point D to the points where circle ω_1 touches the sides of triangle CMD are also equal to x (due to the triangle being equilateral), and distances between the tangent points of any side of triangle CMD with the circles are equal to $2r$. Let distances from vertex A to the tangent points of the sides of the quadrilateral with circle ω_3 be equal to a, and distances from point B to tangent points of the sides of the quadrilateral with ω_3 be equal to b. Then the equality given in (a) can be written as $(x+2r+a)+(x+2r+b)-(a+b)-(2x+2r)=24$; hence $r=12$.

(b) Since $\angle AMB = 60°$, two other angles of triangle ABM add up to $120°$. Consequently, $\angle DAB + \angle CBA = 180° - \angle BAM + {} + 180° - \angle ABM = 240°$. Center of a circle inscribed into an angle belongs to the bisector of this angle, so rays AO and BO are bisectors of angles DAB and CBA respectively. Therefore,

$\angle OAB + \angle OBA = \frac{1}{2}(\angle DAB + \angle CBA) = 120^\circ$, $\angle AOB = $
$= 60^\circ$.

4. Answer: $\pm\frac{1}{3}\sqrt{\sqrt{13}+2}$.

Solution. Both sides of the inequality are nonnegative, and so we can raise them to the second power. We get $(ax-2a)^2 \leqslant x-1$. As this inequality states that the former radicand is greater or equal than some exact square, it is equivalent to the initial one. Combining like terms yields $a^2x^2 - \left(4a^2+1\right)x + \left(4a^2+1\right) \leqslant 0$.

If $a=0$, then $x \geqslant 1$, and that does not satisfy the condition. For all other values of a this inequality is quadratic, the graph of its left side being a parabola that is open up. For negative values of discriminant there are no solutions; if discriminant is zero, there is exactly one solution, and in case discriminant is negative the solution set is a closed interval between the roots of the quadratic function.

Discriminant D is equal to $\left(4a^2+1\right)^2 - 4a^2\left(4a^2+1\right) = 4a^2 +$ $+ 1$. The roots are given by $\frac{4a^2+1\pm\sqrt{D}}{2a^2}$, and the distance between them is $\frac{\sqrt{D}}{a^2}$. This distance has to be equal to 3, hence $\sqrt{D} = 3a^2$, $D = 9a^4$, $9a^4 - 4a^2 - 1 = 0$, $a = \pm\frac{1}{3}\sqrt{\sqrt{13}+2}$.

5. Answer: 18.

Solution. Let w be the number of workers, h be the number of working hours per day, q be the amount of work one worker does in an hour. Let us say that the whole amount of work to be done is equal to 1 (one task). Then $28whq = 1$.

In the second situation we have that the number of workers is $w+2$, they work $h+1$ hours a day, and the task is done in 21 days, therefore, $21(w+2)(h+1)q = 1$.

In the third situation the number of workers is $w+6$, they work $h+2$ hours a day, and complete the task in 15 days, consequently, $15(w+6)(h+2)q = 1$.

Substituting $28whq$ instead of 1 into the right sides of the second and third equations and dividing both parts by a *positive* number q, we get the equations

$$\begin{cases} 21(w+2)(h+1) = 28hw, \\ 15(w+6)(h+2) = 28hw \end{cases}$$

that after expanding and combining like terms look like

$$\begin{cases} hw - 6h - 3w - 6 = 0, \\ 13hw - 90h - 30w - 180 = 0. \end{cases}$$

Subtracting the first equation multiplied by 13 from the second equation yields

$$-12h + 9w - 102 = 0 \iff h = \frac{3w-34}{4}.$$

Then we substitute this expression into the first equation of the latter system:

$$\frac{3w^2-34w}{4} - \frac{9w-102}{2} - 3w - 6 = 0 \iff$$

$$\iff \frac{3}{4}w^2 - 16w + 45 = 0 \iff \left[\begin{array}{l} w = 18, \\ w = \dfrac{10}{3}. \end{array}\right.$$

The value of w has to be integer, therefore $w = 18$.

6. Answer: 4.

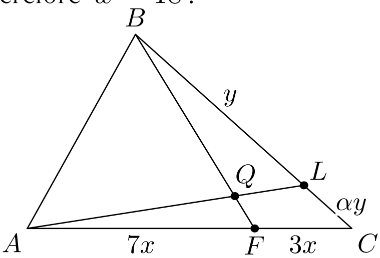

Fig. 3

Solution. Let us designate $CL : LB = \alpha$ (see fig. 3), and area of triangle ABC as S. Menelaus's theorem for triangle ACL and transversal BF yields $\frac{AF}{FC} \cdot \frac{CB}{BL} \cdot \frac{LQ}{QA} = 1$, and from here follows $\frac{7}{3} \cdot (\alpha+1) \cdot \frac{LQ}{QA} = 1$, $\frac{LQ}{QA} = \frac{3}{7\alpha+7}$.

Triangles ALB and ACB have a common altitude dropped from vertex A, and so their areas are in the same ratio as their corresponding bases, i.e. $S_{\triangle ALB} = \frac{S}{\alpha+1}$.

Triangles ALB and QLB have a common altitude dropped from vertex B, hence their areas are also in the same ratio as their bases; consequently $S_{\triangle BLQ} : S_{\triangle BLA} = \frac{QL}{AL} = \frac{3}{7\alpha+10}$. Therefore, $S_{\triangle BLQ} = \frac{3S}{(7\alpha+10)(\alpha+1)}$. It is given that the latter area is equal to $\frac{7S}{36}$ which implies that $7(7\alpha+10)(\alpha+1)=108$, and so $\alpha = \frac{2}{7}$, the second root of this equation being negative.

Ratio of distances from points L and Q to line AC is equal to $LA : QA = (10+7\alpha) : (7+7\alpha) = 4 : 3$. Therefore, the distance in question is $3 \cdot \frac{4}{3} = 4$.

7. Answer: 1524.

Solution. Let $d_1, \ldots, d_6$ be numbers from the first interval, d_7, $\ldots$, d_{12} be numbers from the second interval, $d_{13}, \ldots, d_{18}$ be numbers from the third interval and so on.

Let us notice that each number from the second interval can be represented as $d_i = 30 + c_i$ where $1 \leqslant c_i \leqslant 30$, each number from the third interval can be represented as $d_i = 60 + c_i$ where $1 \leqslant c_i \leqslant 35$ etc. Let us also designate $c_1 = d_1, \ldots, c_6 = d_6$.

Taking in account all the designations above, the sum considered is equal to $6 \cdot (30+60+90) + c_1 + c_2 + \ldots + c_{24} = 1080 + c_1 + c_2 + \ldots + c_{24}$. Let us also notice that all numbers $c_1, c_2, \ldots, c_{24}$ have to be different (if $c_i = c_j$ then difference $d_i - d_j$ is a multiple of 30, and if $c_i \neq c_j$ then $d_i - d_j$ is not a multiple of 30). Therefore, sum of numbers d_i reaches its maximum, if c_i take values from 7 to 30 (in arbitrary order); this maximum is equal to $1080 + 7 + 8 + \ldots + 30 = 1080 + \frac{7+30}{2} \cdot 24 = 1524$.

Grade 10, Problem set #4

1. Answer: $\frac{50}{3}$; $\frac{52}{3}$.

2. Answer: 11242.

3. Answer: (a) 19; (b) 60°.

4. Answer: $\pm\sqrt{\sqrt{5}-2}$.

5. Answer: 12.

6. Answer: 39.

7. Answer: 3122.

Grade 11, Problem set #5

1. Answer: 50; 158.

Solution. We start with finding intersection points of lines with parabola (only positive values of a are considered, since otherwise the parabola does not form a segment on the third line):

$$\begin{cases} y = 2x^2, \\ y = 98 \end{cases} \iff \begin{cases} x = \pm 7, \\ y = 98; \end{cases} \qquad \begin{cases} y = 2x^2, \\ y = 18 \end{cases} \iff \begin{cases} x = \pm 3, \\ y = 18; \end{cases}$$

$$\begin{cases} y = 2x^2, \\ y = a \end{cases} \iff \begin{cases} x = \pm\sqrt{\frac{a}{2}}, \\ y = a. \end{cases}$$

Therefore, lengths of the segments in question are equal to 14, 6 and $\sqrt{2a}$. The angle of 120° is an obtuse angle, so it is the largest angle in a triangle and it lies opposite the largest side of the triangle. Hence two cases are possible.

1) The angle of 120° is opposite the side equal to $\sqrt{2a}$. Then cosine theorem yields $2a = 196 + 36 - 2 \cdot 14 \cdot 6 \cdot \cos 120^\circ$, $a = 158$.
2) The angle of 120° is opposite the side equal to 14. Then cosine theorem yields $196 = 2a + 36 - 2 \cdot \sqrt{2a} \cdot 6 \cdot \cos 120^\circ$, and so $a + 3\sqrt{2}\sqrt{a} - 80 = 0$, $\sqrt{a} = 5\sqrt{2}$ or $\sqrt{a} = -8\sqrt{2}$, hence, $a = 50$.

2. Answer: $g_{\text{min}} = \frac{55}{16}$, $g_{\text{max}} = 5$.

Solution. Let us transform the given function:

$$g(x) = \frac{1}{2}\cos 4x - \frac{1}{2}\cos 10x - \frac{1}{2} + \frac{1}{2}\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 10x + 4 =$$
$$= \frac{1}{2}\left(2\cos^2 2x - 1\right) + \frac{1}{2}\cos 2x + 4 = \cos^2 2x + \frac{1}{2}\cos 2x + \frac{7}{2}.$$

Introducing a new variable $t = \cos 2x$, we get a function $f(t) = t^2 + \frac{1}{2}t + \frac{7}{2}$, $t \in [-1;\ 1]$. The graph of this function is a parabola open up, and its vertex is a point with abscissa $t_0 = -\frac{1}{4}$. Consequently, the smallest value is $f(t_0)$, i.e. $f_{\text{min}} = f\left(-\frac{1}{4}\right) = \frac{55}{16}$. This function reaches its largest value at the point farthest from the vertex of parabola, hence $f_{\text{max}} = f(1) = 5$.

3. Answer: 6132.

Solution. There are two possibilities.

1) The first digit is eight. Then it is followed by six eights, and we have ten last spaces to place zeroes and sevens. Each of these positions can be filled with either a zero or a seven; thus we have 2^{10} ways of doing it, but there are two cases that are not applicable to us (when all ten spaces are occupied by identical digits). All in all, we can obtain $2^{10}-2=1022$ different numbers in this case.
2) The first digit is seven. Then there are ten possibilities to place the eights. After the spaces for them have been chosen, we have to fill the remaining nine positions with zeroes and sevens; at that we must use at least one zero (and one seven is already placed at the first space). This can be done in 2^9-1 different ways (-1 stands for the case when all digits are sevens). Therefore, the total amount is $10\left(2^9-1\right)$.

Summing up the results yields $2^{10}-2+10\cdot 2^9-10=6132$.

4. Answer: (a) 6; (b) 60°; (c) $\dfrac{29\sqrt{3}}{6}$.

Solution. (a) Let lines DA and CB intersect at point M (see fig. 1, p. 32). Triangle CMD is regular, as radii of all three circles are equal to each other. Let us designate radii of the circles as r, and distances from point C to the points where circle ω_2 touches the sides of triangle CMD as x (these distances are equal as segments of tangent lines drawn to a circle from one point). Then distances from point D to the points where circle ω_1 touches the sides of triangle CMD are also equal to x (due to the triangle being equilateral), and distances between the tangent points of any side of triangle CMD with the circles are equal to $2r$. Let distances from vertex A to the tangent points of the sides of the quadrilateral with circle ω_3 be equal to a, and distances from point B to tangent points of the sides of the quadrilateral with ω_3 be equal to b. Then the equality given in (a) can be written as $(x+2r+a)+(x+2r+b)-(a+b)-(2x+2r)=12$; hence $r=6$.

(b) Since $\angle AMB=60^\circ$, two other angles of triangle ABM add up to 120°. Consequently, $\angle DAB+\angle CBA=180^\circ-\angle BAM+$ $+\,180^\circ-\angle ABM=240^\circ$. Center of a circle inscribed into an angle belongs to the bisector of this angle, so rays AO and BO are bisectors of angles DAB and CBA respectively. Therefore, $\angle OAB + \angle OBA = \frac{1}{2}(\angle DAB + \angle CBA) = 120^\circ$, $\angle AOB = = 60^\circ$.

(c) Equating two expressions for the area of triangle OAB, we get $\frac{1}{2} OA \cdot OB \cdot \sin 60^\circ = \frac{1}{2} r \cdot AB$; from this follows that $AB = = \frac{29}{2\sqrt{3}}$.

5. Answer: $\left[-\frac{3}{4}\,;2\right)$.

Solution. Let us make a substitution $\sqrt{x+7} = t$. Then $x = = t^2 - 7$ and the inequality gets the form of

$$\log_{t+7-t^2}\left(t^2-3\right) \geqslant 1 \Leftrightarrow \log_{t+7-t^2}\left(t^2-3\right) \geqslant \log_{t+7-t^2}\left(t+7-t^2\right).$$

Rationalizing[1] both parts yields

$$\begin{cases} \left(t-t^2+6\right)\left(t^2-3-\left(t-t^2+7\right)\right) \geqslant 0, \\ t-t^2+7>0, \\ t-t^2+7 \neq 1, \\ t^2-3>0. \end{cases}$$

The first inequality of this system is equivalent to the following: $\left(t^2-t-6\right)\left(2t^2-t-10\right) \leqslant 0$, $(t+2)^2(t-3)(2t-5) \leqslant 0$, and so $t \in \{-2\} \cup \left[\frac{5}{2}\,;3\right]$.

The second inequality yields $\frac{1-\sqrt{29}}{2} < t < \frac{1+\sqrt{29}}{2}$; the third yields $t \neq -2$ and $t \neq 3$; and the last means that $|t| > \sqrt{3}$.

Intersecting all the sets obtained, we get $t \in \left[\frac{5}{2}\,;3\right)$. Consequently, $\frac{5}{2} \leqslant \sqrt{x+7} < 3 \iff \frac{25}{4} \leqslant x+7 < 9 \iff$ $\iff -\frac{3}{4} \leqslant x < 2$.

6. Answer: 16.

[1] We use the fact that if both expressions $\log_a b - \log_a c$ and $(a-1)(b-c)$ exist, they have the same sign (i.e. they are either both positive or both negative or both zeroes). That allows us to replace an inequality $\log_a b - \log_a c > 0$ by an inequality $(a-1)(b-c) > 0$, taking into account the domain of functions in the initial inequality.

Solution. Let us designate $CL : LB = \alpha$ (see fig. 4), and area of triangle ABC as S. Menelaus's theorem for triangle ACL and transversal BF yields $\frac{AF}{FC} \cdot \frac{CB}{BL} \cdot \frac{LQ}{QA} = 1$, and from here follows $\frac{2}{5} \cdot (\alpha + 1) \cdot \frac{LQ}{QA} = 1$, $\frac{LQ}{QA} = \frac{5}{2\alpha + 2}$.

Triangles ALB and ACB have a common altitude dropped from vertex A, and so their areas are in the same ratio as their corresponding bases, i.e. $S_{\triangle ALB} = \frac{S}{\alpha + 1}$.

Triangles ALB and QLB have a common altitude dropped from vertex B, hence their areas are also in the same ratio as their bases; consequently $S_{\triangle BLQ} : S_{\triangle BLA} = \frac{QL}{AL} = \frac{5}{2\alpha + 7}$.

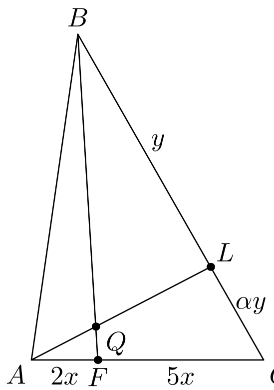

Fig. 4

Therefore, $S_{\triangle BLQ} = \frac{5S}{(2\alpha + 7)(\alpha + 1)}$. It is given that the latter area is equal to $\frac{5S}{12}$ which implies that $(2\alpha + 7)(\alpha + 1) = 12$, and so $\alpha = \frac{1}{2}$, the second root of this equation being negative.

Ratio of distances from points L and Q to line AC is equal to $LA : QA = (7 + 2\alpha) : (2 + 2\alpha) = 8 : 3$. Therefore, the distance in question is $6 \cdot \frac{8}{3} = 16$.

7. Answer: 3165.

Solution. Let $d_1, \ldots, d_6$ be numbers from the first interval, d_7, $\ldots$, d_{12} be numbers from the second interval, $d_{13}, \ldots, d_{18}$ be numbers from the third interval and so on.

Let us notice that each number from the second interval can be represented as $d_i = 45 + c_i$ where $1 \leqslant c_i \leqslant 45$, each number from the third interval can be represented as $d_i = 90 + c_i$ where $1 \leqslant c_i \leqslant 45$ etc. Let us also designate $c_1 = d_1, \ldots, c_6 = d_6$.

Taking in account all the designations above, the sum considered is equal to $6 \cdot (45 + 90 + 135 + 180) + c_1 + c_2 + \ldots + c_{30} = 2700 + c_1 + c_2 + \ldots + c_{30}$. Let us also notice that all numbers $c_1, c_2, \ldots, c_{30}$

have to be different (if $c_i = c_j$ then difference $d_i - d_j$ is a multiple of 45, and if $c_i \neq c_j$ then $d_i - d_j$ is not a multiple of 45). Therefore, sum of numbers d_i reaches its minimum, if c_i take values from 1 to 30 (in arbitrary order); this minimum is equal to $2700 + 1 + 2 + \ldots + 30 = 2700 + \frac{1+30}{2} \cdot 30 = 3165$.

Grade 11, Problem set #6

1. Answer: 49; 337.

2. Answer: $g_{\min} = -\frac{73}{16}$, $g_{\max} = -3$.

3. Answer: 28658.

4. Answer: (a) 5; (b) $60°$; (c) $\frac{21\sqrt{3}}{5}$.

5. Answer: $[-2; 1)$.

6. Answer: 12.

7. Answer: 2075.

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